3) $2n_l = \pm n_g$ from (6.2).

The quantities a, ω must satisfy, for these solutions, the conditions

$$\langle\Phi\rangle + \frac{\partial\langle K_2\rangle}{\partial\omega} = 0, \quad \frac{\partial^2 K_0}{\partial a^2} \neq 0, \quad \frac{\partial\langle\Phi\rangle}{\partial\omega} + \frac{\partial^2\langle K_2\rangle}{\partial\omega^2} \neq 0, \quad q_1 n_l = q_2 n_g \tag{6.3}$$

$$\langle\Phi\rangle \equiv \left\langle \frac{\partial^2 K_1}{\partial a\,\partial\omega} \left\{\int \frac{\partial K_1}{\partial\omega}\,dt\right\}\right\rangle - \left\langle \frac{\partial^2 K_1}{\partial\omega^2} \left\{\int \frac{\partial K_1}{\partial a}\,dt\right\}\right\rangle -$$
$$\left\langle \frac{\partial^2 K_1}{\partial\omega^2}\frac{\partial^2 K_0}{\partial a^2} \left\{\int\left\{\int \frac{\partial K_1}{\partial\omega}\,dt\right\}dt\right\}\right\rangle$$

Analysis of conditions (6.3) yields positive results with regard to the problem of the existence of periodic solutions of the problem for sufficiently small μ in the case of the commensurabilities 1)-3). For example, in the case of commensurability $n_l = 2n_g$ we have

$$\langle\Phi\rangle + \frac{\partial\langle K_2\rangle}{\partial\omega} = \Phi_{1,2}(\theta, \rho, \varphi)\sin(\omega+\lambda)$$
$$\Phi_{1.2}(\theta, \rho, \varphi) = D\sin 2\varphi\,[\sin^2\theta - \sin^2\rho\,(1 + \cos\theta + 2\cos^2\theta)]$$

and conditions (6.3) hold when $\omega + \lambda = 0, \pi$; $\Phi_{1.2}(0, \rho, \varphi) \neq 0$ (here $\cos\theta = a/K_0(a)$, $\cos\rho = H/K_0(a)$, H is an arbitrary constant and $D = \text{const} \neq 0$, φ, λ are the coordinates of the centre of mass of the body in the fixed coordinate system /3/). We have analogous formulas and arguments in the case of the commensurabilities 2), 3).

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Translated by L.K.

PMM U.S.S.R., Vol.51, No.2, pp.189-196, 1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

AVERAGING IN A QUASILINEAR SYSTEM WITH A STRONGLY VARYING FREQUENCY*

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The problem of the applicability of asymptotic averaging methods to single-frequency quasilinear systems are studied for the critical case. It is assumed that in the asymptotically large time interval under consideration the frequency (the derivative of the oscillation or rotational phase) is a slowly varying parameter allowing the singularity to be approximated by a power function of slow time or of a small parameter. The value of the frequence can vary strongly, can become arbitrarily small and equal to zero, and the "frequency" can even change its sign. Such situations arise when studying the oscillating and rotating systems, and particularly often in the problem of the control of specified objects /1/. The present

**Prikl.Matem.Mekhan.*, 51, 2, 244-252, 1987

paper gives an estimate for the error of the averaging method (in the class of power-type estimates in terms of the small parameter, in an asymptotically large time interval). Examples of the analysis of specific mechanical systems are discussed. Non-linear oscillating systems have been studied using the method of averaging in the critical cases("passage" through the separatrix and resonances), in a number of papers (see /2-5/ et al.).

1. Formulation of the problem. Consider a class of quasilinear rotational-oscillatory systems which can be described using Van der Pohl-type variables, by the Cauchy problem of the form /1, 6-8/

$$\begin{aligned} &z^{\cdot} = \varepsilon Z(\tau, z, \varphi),\ z(t_0) = z^\circ,\ \varepsilon \in (0, \varepsilon_0] \\ &\varphi^{\cdot} = \nu(\tau, \varepsilon),\ \varphi(t_0) = \varphi^\circ,\ t \in [t_0, \Theta\varepsilon^{-1}],\ \tau = \varepsilon t \end{aligned} \tag{1.1}$$

Here z is a vector of any dimension, $n \geqslant 1$, $z \in D$ where D is an open space, φ is a scalar phase, $|\varphi| < \infty$, $\tau \in [\tau_0, \Theta]$ is the slow time, $\Theta > 0$ is a constant independent of the small parameter, $\varepsilon > 0$. The function Z is assumed to be a 2π-periodic (or quasiperiodic, or uniformly almost periodic /9/, see Sect.4) function of the phase φ, sufficiently smooth in z, τ in the region under consideration. The parameters of the problem $t_0, z^\circ, \varphi^\circ$ are the known initial data independent of ε.

Under the conditions that the frequency ν is separated from zero ($|\nu(\tau, \varepsilon)| \geqslant \nu_0 > 0$) and the function Z is smooth, the method of averaging enables us to place in 1:1 correspondence with the Cauchy problem (1.1) the problem averaged over φ, whose solution is ε-close to the exact solution /1, 6-8/. Studying the averaged system using analytic or numerical methods is, as a rule, much simpler. The approximate solution obtained can be used to devise constructive procedures for separating the variables z, τ, and φ to an arbitrarily prescribed degree of accuracy in ε and $t \in [t_0, \Theta\varepsilon^{-1}]$, see /1, 6-8/, and a scheme of successive approximations (Picard's method) /10/. From the point of view of practical applications, the study of the critical cases is of interest, when the frequency $\nu(\tau, \varepsilon)$ can vary strongly and pass through asymptotically small values.

We further assume that the frequency $\nu = \nu(\tau, \varepsilon)$ becomes asymptotically small in ε in the time interval in question, or it may become zero, may pass through these values and may even change its sign. To be specific we assume that the function $\nu(\tau, \varepsilon)$ is approximated by one of the expressions of the form

$$\nu = \nu(\tau, \varepsilon) = \gamma(\tau) \left| \frac{\tau_* - \tau}{\Theta - \tau_0} \right|^\alpha + \varepsilon^\beta \omega(\tau, \varepsilon) \tag{1.2}$$

$$\nu = \nu(\tau, \varepsilon) = \gamma(\tau) \left| \frac{\tau_* - \tau}{\Theta - \tau_0} \right|^\alpha \operatorname{sign}(\tau_* - \tau) + \varepsilon^\beta \omega(\tau, \varepsilon) \tag{1.3}$$

The functions $\gamma(\tau)$, $\omega(\tau, \varepsilon)$ and parameters τ_*, α, β appearing in representations (1.2) and (1.3) satisfy the conditions

$$\begin{aligned} &\gamma(\tau) \geqslant \gamma_0 > 0, \quad |\omega(\tau, \varepsilon)| \geqslant \omega_0 > 0 \\ &\tau_* \in [\tau_0, \Theta], \quad 0 \leqslant \alpha < \infty, \quad 0 < \beta \leqslant 1 \end{aligned} \tag{1.4}$$

Here the parameters α, β, τ_* are constants independent of ε. It should be noted that the case of $\beta \geqslant 1$ is equivalent to system (1.1)-(1.3) with $\omega \equiv 0$. This is achieved by introducing an additional slow variable z_{n+1}, $z_{n+1} = \varepsilon^\beta \omega$ and the corresponding phase $\psi = \varphi - z_{n+1}$. Introducing further the dimensionless time

$$\begin{aligned} &t' = \frac{t - t_0}{\Theta - \tau_0}, \quad t' \in [0, \varepsilon^{-1}], \quad \tau' = \varepsilon t' = \frac{\tau - \tau_0}{\Theta - \tau_0}, \quad \tau' \in [0, 1] \\ &\tau_*' = \frac{\tau_* - \tau_0}{\Theta - \tau_0}, \quad \tau_*' \in [0, 1], \quad |\tau_*' - \tau'| \in [0, 1] \end{aligned} \tag{1.5}$$

we reduce the Cauchy problem (1.1) and expressions (1.2)-(1.4) to a more convenient form (the primes are omitted for brevity).

Our aim is to construct a scheme for the approximate solution of the Cauchy problem (1.1)-(1.4) (taking into account the substitution (1.5)), and to substantiate the estimate of the method of averaging in the class of the power estimates in ε an asymptotically large time interval.

The cases when the frequency $\nu(\tau, \varepsilon)$ (1.2) or (1.3) is asymptotically small in ε or when it becomes vanishingly small in the process of the evolution of the system, substantially complicate the application of various averaging schemes /1, 4-8/. The difficulties may be caused by the special features of the right-hand sides of the standard-type equations (see

Sect.3), and by the failure of the basic requirement of the averaging method as regards the existence of their uniform means in t /6-8/. Proofs of the validity of the averaging schemes based on the corresponding changes of the slow variables and estimations of the error of the method become unsound, since the expressions in the formulas for the change of variable contain the frequency in the denominator /1, 6-8/. Analysis of elementary examples /5/ shows that when the frequency is asymptotically small $(\nu \sim \varepsilon)$, the solutions of the formally phase-averaged system and of the initial system are not, as a rule, close to each other when $t \sim \varepsilon^{-1}$. Thus situations are possible in which the method of averaging is essentially unsuitable for approximate investigations of systems of the type (1.1)-(1.4) (φ is the "slow" or "non-rotating" phase). In such cases other asymptotic or numerical-analytic methods should be used. In the case of the class of systems (1.1)-(1.4) in question (we have a number of other, fairly wide classes of rotational-oscillatory systems) the method of averaging is found to be formally inapplicable, therefore we need more accurate estimates of the error, and these may be satisfactory when solving applied problems.

2. Constructing estimates for the method of averaging. Below, we shall consider special cases of expressions (1.2)-(1.4) for $\beta = 1$ which, as was shown above, is equivalent to the case $\omega \equiv 0$. Thus in order to be specific, let us consider the function ν of the form (1.2) with $\omega \equiv 0$, i.e. in the dimensionless variables (1.5)

$$\nu = \nu(\tau) = \gamma(\tau) \mid \tau_* - \tau \mid^{\alpha}, \quad \alpha \geqslant 0, \quad \tau, \tau_* \in [0, 1] \tag{2.1}$$

Then we can write in (1.1), (2.1), without loss of generality, $\gamma(\tau) = \gamma_* = \mathrm{const}$. This is achieved by replacing the argument τ by ϑ according to the formulas

$$\gamma_* \mid \vartheta_* - \vartheta \mid^{\alpha} d\vartheta / d\tau = \gamma(\tau) \mid \tau_* - \tau \mid^{\alpha}$$

$$\vartheta = \vartheta(\tau, \tau_*), \quad \vartheta(0, \tau_*) = 0, \quad \vartheta_* = \vartheta(\tau_*, \tau_*)$$

$$\mid \vartheta_* - \vartheta \mid^{1+\alpha} \operatorname{sign}(\vartheta_* - \vartheta) = \frac{1+\alpha}{\gamma_*} \int_{\tau_*}^{\tau} \gamma(\lambda) \mid \tau_* - \lambda \mid^{\alpha} d\lambda$$

$$\vartheta_* = \left[\frac{1+\alpha}{\gamma_*} \int_0^{\tau_*} \gamma(\tau) \mid \tau_* - \tau \mid^{\alpha} d\tau\right]^{1/(1+\alpha)}$$

Therefore let the known function $\zeta = \zeta(\tau, z^\circ)$ be a solution of system (1.1) formally averaged over φ for z:

$$\zeta' = Z_0(\tau, \zeta), \quad \zeta(0) = z^\circ, \quad (') \equiv (d / d\tau) \tag{2.2}$$

$$Z_0(\tau, \zeta) = \frac{1}{2\pi} \int_0^{2\pi} Z(\tau, \zeta, \varphi) \, d\varphi, \quad \tau \in [0, 1], \quad \zeta \in D$$

Then the estimate of the error $(z - \zeta)$ is obtained, as usual /6-8/, using Gronwall's lemma. Using the subsitution

$$z = \zeta + v + \delta, \quad \varphi(t, \varepsilon) \equiv \varphi^\circ + \int_0^t \nu(\varepsilon s) \, ds \tag{2.3}$$

$$v = v(t, \tau, \zeta, \varepsilon) \equiv \varepsilon \int_0^t [Z(\tau, \zeta, \varphi(s, \varepsilon)) - Z_0(\tau, \zeta)] \, ds$$

in which the variables τ, ζ are regarded as parameters during the integration over s, we obtain the following estimate for the unknown δ using standard methods /6/:

$$\mid \delta \mid \leqslant e^L \max_t \varepsilon \int_0^t \left(L \mid v \mid + M \left\| \frac{\partial v}{\partial \zeta} \right\| + \left| \frac{\partial v}{\partial \tau} \right|\right) ds \leqslant \tag{2.4}$$

$$K e^L \max_t \left(\mid v \mid + \left\| \frac{\partial v}{\partial \zeta} \right\| + \left| \frac{\partial v}{\partial \tau} \right|\right), \quad t \in [0, \varepsilon^{-1}]$$

Here L is the Lipshitz constant of the function Z in $z \in D$, M is the maximum value of Z_0 in τ, $\tau \in [0, 1]$, K is a constant determined in terms of L, M. Thus, in accordance with (2.3), (2.4), the error $(z - \zeta)$ of the method of averaging is determined by the estimates of $\mid v \mid$, $\mid \partial v / \partial \tau \mid$ and $\| \partial v / \partial \zeta \|$, which are of the same order of smallness in ε for $t \in [0, \varepsilon^{-1}]$. Below we give a method of obtaining such estimates for system (1.1), (2.1) in question.

Thus, let us require an estimate of the quantity

$$w = \max_t \varepsilon \left| \int_0^t f(\varphi(s, \varepsilon))\, ds \right|, \quad t \in [0, \varepsilon^{-1}] \tag{2.5}$$

in terms of the parameter ε . Here $f(\varphi)$ is a periodic (possibly quasiperiodic, see Sect.4) function of φ, which has zero mean according to (2.3). The differential relation connecting t and φ can be reduced to an expression of the form (we assume here, for brevity, that $\varphi(0) = 0$)

$$\varphi^{\cdot} \equiv d\varphi/dt = \nu(\tau) \equiv \gamma_* |\tau_* - \tau|^\alpha = \tag{2.6}$$
$$\gamma_* |\tau_*^{1+\alpha} - (1+\alpha)\gamma_*^{-1}\sigma|^{\alpha\rho} \equiv \lambda(\sigma), \quad \rho = (1+\alpha)^{-1}, \quad \sigma = \varepsilon\varphi$$

Then, taking (2.6), into account, we can rewrite the expression for w (2.5) as follows:

$$w = \frac{1}{\gamma_*} \max_\varphi \varepsilon \left| \int_0^\varphi f(\psi) \left| \tau_*^{1+\alpha} - \frac{1+\alpha}{\gamma_*}\varepsilon\psi \right|^{-\alpha\rho} d\psi \right| \tag{2.7}$$
$$\varphi \in [0, \Gamma\varepsilon^{-1}], \quad \Gamma = \frac{\gamma_*}{1+\alpha} |\tau_*^{1+\alpha} - |\tau_* - 1|^{1+\alpha}|$$

The value of the new slow variable $\sigma = \sigma_* \equiv \gamma_* (1+\alpha)^{-1}\tau_*^{1+\alpha}$, for which the frequency $\lambda(\sigma_*) = \nu(\tau_*) = 0$, satisfies, according to (2.6), the inequalities $0 \leqslant \sigma_* \leqslant \Gamma$, since $0 \leqslant \tau_* \leqslant 1$. Furthermore, using elementary transformations and substitutions, we can obtain the following expression for the required estimate w (2.7):

$$w \leqslant O\varepsilon^\rho, \ 0 < \rho \leqslant 1, \ O = \text{const} \tag{2.8}$$
$$O = \gamma_*^{-\rho}\rho^{\alpha\rho} [|F(\sigma_*/\varepsilon)| + \max_{\sigma \in [\sigma_*, \Gamma]} |F((\sigma - \sigma_*)/\varepsilon)|]$$
$$F(\varkappa/\varepsilon) = \int_0^{\varkappa/\varepsilon} \frac{f(\sigma_*/\varepsilon - y)}{y^{\alpha\rho}}\, dy$$

Since $f(\sigma_*/\varepsilon - y)$ is a 2π-periodic (or quasiperiodic) function of y which has uniform zero mean with respect to ε, and the index $\alpha\rho$ satisfies the inequalities $0 \leqslant \alpha\rho < 1$, it follows that we can obtain uniform estimates for the integrals in (2.8).

According to (2.8), we need to estimate the improper integral $F(\varkappa/\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$, $\varkappa \in [0, \Gamma]$. The integrand has a singularity at $y = 0$ and the upper limit of the integrand $\varkappa/\varepsilon \to \infty$ as $\varepsilon \to 0$. To obtain the estimate for this integral, we separate it into two integrals

$$F(\varkappa/\varepsilon) = F(h) + (F(\varkappa/\varepsilon) - F(h)) \equiv F_1 + F_2 \tag{2.9}$$
$$|F| \leqslant |F_1| + |F_2|, \quad 0 \leqslant h < \varkappa/\varepsilon$$
$$|F_1| \leqslant ah^\rho, \quad a = (1+\alpha) \max_{|y|<\infty} |f|$$
$$|F_2| \leqslant b(\varepsilon)[h^{-\alpha\rho} + 2(\varepsilon/\varkappa)^{\alpha\rho}]$$
$$b(\varepsilon) = \max_{0 \leqslant h \leqslant H \leqslant \varkappa/\varepsilon} \left| \int_h^H f\, dy \right|$$

The estimate for F (2.9) reaches its minimum with respect to h $(0 \leqslant h \leqslant \varkappa/\varepsilon)$ when $h = \alpha b/a$. As a result we can obtain the following estimate uniform in ε:

$$|F(\varkappa/\varepsilon)| \leqslant a^{\alpha\rho}\alpha^{-\alpha\rho}b^\rho/\rho + 2b(\varepsilon)(\varepsilon/\varkappa)^{\alpha\rho} \tag{2.10}$$

Thus the required estimate of the error in the method of averaging for system (1.1), (2.1) averaged over φ, is reduced, by virtue of expressions (2.3), (2.4) and (2.10), to the form

$$|z - \zeta| \leqslant Ce^L\varepsilon^\rho, \ t \in [0, \varepsilon^{-1}] \tag{2.11}$$
$$\varepsilon \in (0, \varepsilon_0], \quad 0 < \rho \leqslant 1 \quad (0 \leqslant \alpha < \infty)$$

Here C is a constant which can be determined constructively from the parameters K, γ_*, α introduced above and the constants a, b, characterizing, in accordance with (2.3), (2.5), (2.8)-(2.10), the estimates of $|v|$, $|\partial v/\partial\tau|$, $\|\partial v/\partial\zeta\|$. Under the assumptions made in Sect.1 with regard to the properties of smoothness of the vector function Z (and especially about the continuous differentiability in z, τ and continuity in φ) the function f corresponding, respectively, to v, $\partial v/\partial\tau$, $\partial v/\partial\zeta$ in the estimate (2.4), will be uniformly bounded for all values of the phase φ, discussed here. This leads to the boundedness of the corresponding coefficients $a(\alpha)$ $(0 \leqslant \alpha < \infty)$ in estimate (2.9) for F_1. The mean value of the periodic or quasiperiodic function, f i.e. possessing a finite-dimensional frequency basis (see /11/), is by definition equal to zero. Therefore, the integral in the corresponding expression $b(\varepsilon)$ of estimate (2.9) for F_2 is also bounded. This follows directly from the Fourier representation /5-8, 11/. When the function $f(\varphi)$ $(|\varphi| < \infty)$ is uniform and almost periodic, we will need the condition that the set of frequencies λ_k of its Fourier representation /9/ is separated from zero, i.e.

$$\lambda_k \geqslant \lambda^* > 0, \quad k = 1, 2, \ldots; \quad f(\varphi) = \sum_{k=1}^{\infty} (f_k^c \cos \lambda_k \varphi + f_k^s \sin \lambda_k \varphi)$$

Otherwise, when $\lambda_k \downarrow 0$ and $k \to \infty$, we will assume that the coefficients f_k^c, f_k^s decrease fairly rapidly. For example, we obtain the function f in the form shown, when system (1.1) is is linear in z, and has an uniformly almost periodic matrix of the coefficients and the inhomogeneity vector /9/.

From (2.11) there follows the estimate $|z - \zeta| = O(\varepsilon)$ when $\alpha = 0$, which corresponds to the equality $\nu = \gamma$, i.e. to the non-critical case /6-8/. In the limit, as $\alpha \to \infty$, we find that $F \to \infty$. However, because of expression (2.8) for the coefficient O. The coefficient C in (2.11) will be bounded, i.e. the estimate $|z - \zeta| = O(1)$ corresponding to the case of $\nu = O(\varepsilon)$, and in particular to $\nu \equiv 0$ /5/, will hold. We note that the estimate (2.11) remains valid in more general cases when the function γ depends continuously on ε $(\gamma = \gamma(\tau, \varepsilon))$ and the right-hand side of system (1.1) for z also depends on ε, and a relation of the form

$$Z = Z^\circ(\tau, z, \varphi) + \varepsilon^\rho Z^*(\tau, z, \varphi, \varepsilon), \quad |Z^*| \leqslant M$$
$$z \in D, \quad \tau \in [0, 1], \quad |\varphi| < \infty, \quad \varepsilon \in (0, \varepsilon_0]$$

is allowed.

Using relations of the type (2.3)-(2.10) and taking into account the non-unique relationship $t(\varphi)$ or $\tau(\sigma)$ (see (2.5)-(2.8)), we establish the estimate (2.11) for the case of a sign-variable frequency ν (1.3) when $\omega \equiv 0$. Analogous estimates hold in the cases when the frequency can pass repeatedly through zero values. The situations in which the frequency $\nu(\tau, \varepsilon)$ is a discontinuous function of τ, require a separate study.

Let us consider an additional case when we have $\nu = O(\varepsilon^\beta)$, $0 \leqslant \beta < 1$, in the time interval $t \in [0, \varepsilon^{-1}]$ in question. This corresponds to $\gamma(\tau) \equiv 0$ in expressions (1.2) and (1.3). Then, introducing a new small parameter $\mu = \varepsilon^{1-\beta}$ and the argument $\theta = \varepsilon^\beta t \in [0, \mu^{-1}]$ and taking into account (1.5), we reduce system (1.1)-(1.4) to the standard form. We can employ in this system the usual procedure of averaging over the rapid "in slow time" θ phase φ in accordance with /6-8/. When the demands that the function Z be smooth with respect to the slow variable given above are met, the first-order scheme of averaging leads to an error of the order of $O(\mu) = O(\varepsilon^{1-\beta})$ for $\theta \sim \mu^{-1}$ or $t \sim \varepsilon^{-1}$.

Combining all the above examples, we can estimate the error of the method of averaging for a more general situation (see (1.2)-(1.4), (2.1))

$$\nu = \nu(\tau, \varepsilon) \equiv \varepsilon^\eta \gamma_0(\tau) \left| \frac{\tau_* - \tau}{\theta - \tau_0} \right|^\alpha, \quad 0 \leqslant \eta < 1, \quad 0 \leqslant \alpha < \infty \tag{2.12}$$

$$\nu = \nu(\tau, \varepsilon) \equiv \varepsilon^\eta \gamma_0(\tau) \left| \frac{\tau_* - \tau}{\theta - \tau_0} \right|^\alpha \operatorname{sign}(\tau_* - \tau) \tag{2.13}$$

In this case the estimate of the error of the method of averaging based on expressions (2.11) and the estimates for the additional case mentioned above, are written as follows:

$$|z - \zeta| \leqslant C\mu^\rho = C\varepsilon^\chi, \quad \chi = (1 - \eta)(1 + \alpha)^{-1} \tag{2.14}$$
$$0 < \chi \leqslant 1, \; t \in [t_0, \Theta\varepsilon^{-1}], \; \theta \in [\theta_0, \Theta\mu^{-1}], \; C = \text{const}$$

If the expression for the frequency $\nu(\tau, \varepsilon)$ is of the usual form, e.g. (1.2) where $\gamma(\tau)$ and $\omega(\tau, \varepsilon)$ are sign-definite functions of the same sign, then the solution $\zeta(\tau, z^\circ)$ of system (1.1) averaged over φ for $t - t_0 \sim \varepsilon^{-1}$ differs from the exact solution $z(t, z^\circ, \varepsilon)$ by an amount of the order of $O(\varepsilon^\delta)$ $(\delta \geqslant 0)$, with $\delta = \max(\rho, 1 - \beta)$ in the case when $\gamma(\tau, \varepsilon) = O(1)$ or $\delta = \max(\chi, 1 - \beta)$, provided that $\gamma(\tau, \varepsilon) = O(\varepsilon^\eta)$, as in (2.12). In the general case, when the expressions $\nu(\tau, \varepsilon)$ have the form (1.3) (and of the type (2.13)), we must investigate in more detail the process of "passage" through the zero value of the frequency and of its possible "sticking" in the neighbourhood of asymptotically small values.

3. Examples of the dependence of the standard system and the frequency on the small parameter for specific oscillatory systems. An almost free system.

We shall consider a vector system of the form

$$x^{\cdot\cdot} = \varepsilon^2 f(\tau, x, \varepsilon^{-1}x^\cdot, \varphi, \varepsilon), \; x(0) = x^\circ, \; x^\cdot(0) = \varepsilon v^\circ \tag{3.1}$$
$$\varphi^\cdot = \nu(\tau, \varepsilon), \; \varphi(0) = \varphi^\circ; \; \tau = \varepsilon t \in [0, \Theta], \; \varepsilon \in (0, \varepsilon_0]$$

Here $x, x^\cdot = dx/dt$ are the n-vectors, φ is the scalar phase, and $f(\tau, x, u, \varphi, \varepsilon)$ is a regular vector function. Using the substitution $x^\cdot = \varepsilon y, z = (x, y)$, we reduce the Cauchy problem for system (3.1) to its standard form (1.1) in which the right-hand side $\varepsilon Z = (\varepsilon y, \varepsilon f(\tau, x, y, \varphi, \varepsilon))$. Making the corresponding assumption about the properties of the functions f and ν, we can apply to it the procedures of the averaging method for $t \in [0, \Theta\varepsilon^{-1}]$, developed and proved above.

In particular, if $f = f(\tau, x, x^{\cdot}, \varphi, \varepsilon)$, then the substitution $x^{\cdot} = \varepsilon y$ results in the independence of this function of y in the first approximation in ε and $t \sim \varepsilon^{-1}$.

A quasilinear scalar oscillatory system of the form $x^{\cdot\cdot} + \nu^2 x = \varepsilon \nu f$ with initial conditions $x(0) = x^\circ, x^{\cdot}(0) = \nu(0, \varepsilon) v^\circ$, can be reduced with help of a Van der Pohl-type substitution to the form (1.1) corresponding to the additional case, provided that $\nu = \varepsilon^\beta \omega(\tau, \varepsilon)$.

A perturbed system with small gyroscopic forces. For example, an equatorial component of the angular velocity vector of perturbed rotations of a dynamically symmetrical body is described by a Cauchy problem of the form /1/

$$x^{\cdot} = \nu y + \varepsilon f(\tau, x, y, \varepsilon),\ x(0) = x^\circ$$
$$y^{\cdot} = -\nu y + \varepsilon g(\tau, x, y, \varepsilon),\ y(0) = y^\circ \tag{3.2}$$

Carrying out the Van der Pohl-type subsitution of (x, y) by (a, b, φ)

$$x = a\cos\varphi + b\sin\varphi,\ y = -a\sin\varphi + b\cos\varphi,\ \varphi^{\cdot} = \nu(\tau, \varepsilon)$$

we can obtain from (3.2) the equivalent standard problem

$$a^{\cdot} = \varepsilon(f\cos\varphi - g\sin\varphi),\ a(0) = a^\circ = x^\circ$$
$$b^{\cdot} = \varepsilon(f\sin\varphi + g\cos\varphi),\ b(0) = b^\circ = y^\circ \tag{3.3}$$

Making the corresponding assumption about the properties of smoothness of the functions f, g and the structure of the functions $\nu(\tau, \varepsilon)$ (see (1.2), (1.3), et al.), we can apply various schemes of the method of averaging to Cauchy problem (3.3). We should note that the dependence of the frequency ν on τ can result from the variation in the axial component of the angular velocity vector of the rigid body (see Sect.4). Other cases of a mechanical character are possible, reducible to the type (1.1) of the quasilinear rotational-oscillatory systems studied by asymptotic methods as in Sect.2.

4. Optimal stabilization of the axial rotation of a dynamically symmetric apparatus by means of small controlling force moments. We study the problem of quenching the equatorial component of the vector $\boldsymbol{\omega}$ of angular velocity of a rigid body, when the variable rate of axial rotation ω_3 is given. The equations of controlled rotations in the coordinate system attached to the principal (central) axes of inertia and the corresponding boundary value problem take the form /1/

$$I\omega_1^{\cdot} + (I_3 - I)\,\omega_3\omega_2 = M_1,\ \omega_1(0) = \omega_1^\circ,\ \omega_1(T) = 0$$
$$I\omega_2^{\cdot} - (I_3 - I)\,\omega_3\omega_1 = M_2,\ \omega_2(0) = \omega_2^\circ,\ \omega_2(T) = 0 \tag{4.1}$$
$$I_3\omega_3^{\cdot} = M_3,\ \omega_3(0) = \omega_3^\circ,\ \omega_3(T) = \omega_3^T$$

The inertia tensor $J = \operatorname{diag}(I_1, I_2, I_3)$ in the problem in question satisfies the condition $I_1 = I_2 = I \neq I_3$ (see (4.1)). Furthermore we introduce, for convenience, the dimensionless variables: the time t', phase coordinates $l_{1,2}$ and ν, the controls $u_{1,2}, v$, a small parameter ε, and write down the corresponding initial and final values

$$t' = \omega^\circ t,\ l_{1,2} = I\omega_{1,2} / L^\circ,\ \nu = (I_3\omega_3 / L^\circ)\,N$$
$$\varepsilon\alpha_{1,2}u_{1,2} = M_{1,2} / (L^\circ\omega^\circ),\ \varepsilon v = (M_3 / (L^\circ\omega^0))\,N,\ T' = \omega^\circ T \tag{4.2}$$
$$l_{1,2}(0) = I\,\omega_{1,2}^\circ / L^\circ = l_{1,2}^\circ,\ l_{1,2}(T') = 0$$
$$\nu\,|_{t'=0,T'} = (I_3\omega_3^{\circ,T} / L^\circ)\,N = \nu^{\circ,T},\quad N = (I_3 - I)\,L^\circ/(II_3\omega^\circ)$$

Here $\varepsilon\alpha_{1,2}$ $(\varepsilon \ll 1)$ are small quantities characterizing the effectiveness of the controls ω°, L° are the initial values of the angular velocity vector $\boldsymbol{\omega}$ and kinetic moment $\mathbf{L} = J\boldsymbol{\omega}$ respectively, and the primes accompanying t' and T' will be omitted from now on. Using relations (4.2), we reduce the boundary value problem (4.1) to the form (see (3.2))

$$l_{1,2}^{\cdot} \pm \nu l_{2,1} = \varepsilon\alpha_{1,2}u_{1,2},\quad l_{1,2}(0) = \overset{\circ}{l}_{1,2},\quad l_{1,2}(T) = 0$$
$$\nu^{\cdot} = \varepsilon v,\ \nu(0) = \nu^\circ,\ \nu(T) = \nu^T \tag{4.3}$$

We further assume, that the instant of time $t = T$ of termination of the control process is determined by varying the magnitude of $\nu(t)$, characterizing the angular velocity ω_3 of the axial twist of the apparatus. This may depend on the constraints imposed on the control v, $|v| \leqslant v_0$ (i.e. on the axial force moment M_3) and on the essential quantity $|\nu^T - \nu^\circ|$ (i.e. on the difference $(\omega_3^T - \omega_3^\circ)$). Thus the control $v = v^*(t)$, the variable $\nu = \nu(\tau)$ and time T are as follows:

$$v = v^*(t) = v_0\operatorname{sign}(\nu^T - \nu^\circ),\ \nu = \nu(\tau) = \nu^\circ + v^*\tau$$
$$\tau = \varepsilon t \in [0, \Theta],\ \Theta = \varepsilon T = |\nu^T - \nu^\circ| / v_0\ \ (v^* \equiv 0,\ t > T) \tag{4.4}$$

Comparing with the results of /1/, we consider here a more general case, when the axial component of the angular velocity may change its sign, i.e. when there exists $\tau = \tau_* \in (0, \Theta)$ such that $\nu(\tau_*) = 0$. We pose the problem of control, optimal with respect to the energy consumed, over the variable $l_{1,2}$, with the instant T of the termination of the process fixed according to (4.4).

We introduce, in accordance with Sect.3, the osculating variables $a_{1,2}$ by means of the variable substitution $l_1 = a_1 \cos\varphi - a_2 \sin\varphi$, $l_2 = a_1 \sin\varphi + a_2 \cos\varphi$, $\varphi^\cdot = \nu$, where the phase φ is a known function of time. As a result of the transformation, the problem of optimal control in question is reduced to the standard form /1/

$$\begin{aligned}
&a_1^\cdot = \varepsilon(\alpha_1 u_1 \cos\varphi + \alpha_2 u_2 \sin\varphi),\quad a_1(0) = l_1^\circ,\quad a_1(T) = 0 \\
&a_2^\cdot = \varepsilon(-\alpha_1 u_1 \sin\varphi + \alpha_2 u_2 \cos\varphi),\quad a_2(0) = l_2^\circ,\quad a_2(T) = 0 \\
&\varphi^\cdot = \nu(\tau) \equiv \nu^\circ + v^*\tau,\quad \varphi(0) = 0 \\
&\Phi[u_1, u_2] = \frac{1}{2}\int_0^\Theta (u_1^2 + u_2^2)\,d\tau \to \min_{u_1, u_2}
\end{aligned} \tag{4.5}$$

Here we assume that the quantity Θ is sufficiently large (see above), therefore the possible constraints which can be imposed on the controls u_1, u_2 are not attained /1/ and do not appear in the formulation of the problem (4.5). Using the Pontryagin maximum principle we can establish that the variables p_1, p_2 (moments) conjugated with the phase variables a_1, a_2, are retained, i.e. $p_{1,2} =$ const, and the optimal control is equal to

$$u_1 = \alpha_1(p_1 \cos\varphi - p_2 \sin\varphi),\quad u_2 = \alpha_2(p_1 \sin\varphi + p_2 \cos\varphi) \tag{4.6}$$

Substituting the expressions for u_1, u_2 (4.6) into Eqs.(4.5) and taking the initial conditions into account, we arrive at the special case of the Cauchy problem of the form (1.1), (1.3), since its right-hand sides are known 2π-periodic functions of the phase φ. The solution of the boundary value problem can also be obtained by elementary methods and leads to very bulky expressions. Averaging over φ enables us, according to Sect.2, to obtain simple approximate expressions for the phase variables and moments

$$\begin{aligned}
&\xi_{1,2} = \mathring{l}_{1,2} + {}^1/_2\eta_{1,2}(\alpha_1^2 + \alpha_2^2) = \mathring{l}_{1,2}(1 - \tau\Theta^{-1}) \\
&\eta_{1,2} = -2\mathring{l}_{1,2} / [\Theta(\alpha_1^2 + \alpha_2^2)]
\end{aligned} \tag{4.7}$$

The values of the constants $\eta_{1,2}$ in (4.7) are given by the final zero conditions for $\xi_{1,2}$, and determined by elementary methods. Substituting $\eta_{1,2}$ into the constants $u_{1,2}$ in place of $p_{1,2}$, gives the required, approximately optimal program control (and its synthesis)

$$\begin{aligned}
&u_1^* = -2\alpha_1(\alpha_1^2 + \alpha_2^2)^{-1}\Theta^{-1}(l_1^\circ \cos\varphi - l_2^\circ \sin\varphi) \\
&u_2^* = -2\alpha_2(\alpha_1^2 + \alpha_2^2)^{-1}\Theta^{-1}(l_1^\circ \sin\varphi + l_2^\circ \cos\varphi) \\
&(\mathring{l}_{1,2} \to l_{1,2},\ \Theta \to \Theta - \tau)
\end{aligned} \tag{4.8}$$

The control $u_{1,2}^*(\varphi, l_1^\circ, l_2^\circ)$ (4.8) is approximately optimal in the following sense. The solution of the corresponding Cauchy problem (4.5) leads to the functions $a_{1,2}^*(t, l_1^\circ, l_2^\circ, \varepsilon)$, whose values at $t = T$ are determined in the $\sqrt{\varepsilon}$-neighbourhood of zero. The value of the functional Φ (4.5) for $u_{1,2}^*$ (4.8) differs by an amount of the order of $O(\sqrt{\varepsilon})$ from the exact minimum value, and this is established by direct solution of the initial problem of optimal control (4.5).

It should be noted that the approximate solution (4.4), (4.7), (4.8) of the problem of optimal quenching of the equatorial component of the angular velocity vector of the body, and of bringing its axial component to the $\sqrt{\varepsilon}$-neighbourhood of the prescribed value, i.e.

$$|\nu(\Theta) - \nu^T| \leqslant C\sqrt{\varepsilon},\quad [l_1^2(\Theta) + l_2^2(\Theta)]^{1/2} = [a_1^2(\Theta) + a_2^2(\Theta)]^{1/2} \leqslant C\sqrt{\varepsilon}$$

given above, holds also in the case when the body in question is not strictly dynamically symmetric and the estimates $|I_1 - I_2| = O(\varepsilon)$, $|I_{1,2} - I_3| = O(1)$ hold. The proposed approach can also be used with a wider class of perturbations acting on the system (4.1), depending on the vector ω and slow time τ (see (3.2), (3.3)).

With regard to the practical applications, it is important to develop and extend the analogous approach of the approximation method to the multifrequency systems under the conditions of "passage" and "sticking" at the resonances.

The analysis of the essentially non-linear systems, single phase of the type (1.1) and the multiphase systems for which the frequencies $\nu_j = \nu_j(z)$ may attain, in the course of evolution, or even "pass", zero or resonant values /2-5/, are of great practical interest, at the same time presenting considerable theoretical difficulties.

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Translated by L.K.

PMM U.S.S.R.,Vol.51,No.2,pp.196-201,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

INVESTIGATION OF PARTIAL ASYMPTOTIC STABILITY AND INSTABILITY BASED ON THE LIMITING EQUATIONS*

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A new type of limiting equations is studied, used to investigate the asymptotic stability and instability of unperturbed motion with respect to some of the variables, based on the Lyapunov function with a sing-constant derivative, without assuming that the perturbed motions are bounded over the non-controlled coordinates. Sufficient conditions are derived for the asymptotic stability with respect to the generalized velocities and some of the generalized coordinates of the zero position of equilibrium of the non-autonomous, holonomic and non-holonomic mechanical systems under the action of dissipative forces.

1. Let us consider the following system of equations:

$$x^{\cdot} = X(t, x) \quad (X(t, 0) \equiv 0) \tag{1.1}$$
$$x \in R^m, x = (y, z), \; y \in R^s, \; z \in R^p \; (m = s + p)$$

The function $X(t, x)$: $R^+ \times \Gamma \to R^m$ ($R^+ = [0, +\infty[$, $\Gamma = \{\|y\| < H > 0, \; \|z\| < +\infty\}$, $\|y\|$ is a norm in R^s, $\|z\|$ in R^p, $\|x\| = \|y\| + \|z\|$) satisfies the conditions for the existence of solutions in the Caratheodory sense /1/. A locally integrable function $r(t) \in L_1$ exists, continuous in x for fixed t, measurable in t for fixed x, for every compact set $K \subset \Gamma$ such that $\|X(t, x)\| \leqslant r(t)$. We shall also assume that system (1.1) satisfies the conditions of z-continuability of the solutions /2/.

We will also introduce a shift of the function $X(t, x)$ in t by an amount $\tau \geqslant 0$ according

*Prikl.Matem.Mekhan.,51,2,253-259,1987